On the Optimality of Path-Dependent Structured Funds: the Cost of Standardization

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Abstract

This paper examines the suitability of an important class of standard financial structured products, namely those whose performances are based on smoothing the return of a given risky underlying asset while providing a guarantee at maturity. Using various assumptions about the customers attitudes towards risk, we show that such standardized products are not optimal, even if the financial market volatility is constant. As a by-product, we provide in particular the optimal portfolio value in the regret/rejoice framework to go further with the notion of aversion of getting a return smaller than the risk-free one. Using the notion of compensating variation, we determine for the first time, the monetary losses of providing these standardized products instead of the optimal ones to the customers. We show that these monetary losses can be very significant when the volatility of the risky asset is stochastic. From the operational point of view, such results highly suggest to trade on the Volatility Index (VIX) and/or to introduce derivatives written on it, when selling standardized funds in order to better meet investors needs and preferences.

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1 Introduction

Structured financial products issued by financial institutions have been introduced in order to allow investors to invest on complex financial instruments such as derivatives while satisfying specific needs. Generally, the proposed portfolios provide a capital protection against market fluctuations while allowing to benefit from market rises. They are based on more or less complex options the underlying assets of which can be a single security, a basket of securities or financial indices.

A first problem that arises with structured products is their fair pricing. It has been examined by several authors using comparisons of prices in the primary or secondary market to the theoretical fair values: Chen and Kensinger (1990) and Chen and Sears (1990) deal with the pricing of convex instruments on the S&P 500; Wasserfallen and Schenk (1996) and Burth *et al.* (2001) examine the Swiss market; Stoimenov and Wilkens (2005) investigate the fair pricing of equity-linked structured products in the German private retail banking sector; Bertrand and Prigent (2015a) study the fair pricing of French financial structured products. All these results show that almost all types of equity-linked structured products are priced above their theoretical values, depending on their complexity degree.

The second important problem, which has been less examined in the financial literature, is the suitability of a given structured product to the investor's risk profile (see Shefrin and Statman, 1993; Driessen and Maenhout, 2007; Das and Statman, 2013). Both the U.S. Dodd-Frank Act[1] and the European MiFID directive[2] have emphasized that financial institutions must check the adequacy of structured products to their retail investors, in light of investor financial protection (see Chang *et al.*., 2015). Most of these structured products provide a capital guarantee[3]. Therefore, they are related to portfolio insurance (PI), which has been extensively used by the financial management industry in equities, bonds and hedge funds. One family of such products is devoted to the smoothing of the performance of a given underlying asset over time (typically a financial index or a basket of indices). Two main methods are introduced to reach such goal: one is based on the introduction of an Asian call option which yields part of the positive performance of the average of the benchmark asset values along the management period; the other one is based on an average of past positive performances of the benchmark asset.[4]

Both types of funds allow the portfolio to be less sensitive to the terminal value of the underlying asset. Indeed, due to their averaging property, they exhibit lower volatilities than their underlying assets and take better account of the whole path of the benchmark asset. However, Bertrand and Prigent (2015b) show that complexity of such products does not always gain advantage over simpler products, when using performance measures such as Kappa ratios (that include for instance both Omega and Sortino ratios). Therefore a natural question arises: Are these funds optimal or not according to main decision criteria ? And, if not, how to measure their inadequacy to customers needs ? For this latter purpose, de Palma and Prigent (2008, 2009) introduce the compensating variation approach as a quantitative methodology to provide investment recommendations relying both on the performances of the financial products and on the investor preferences. For the standard allocation problem on simple assets such as equities, bonds and

[1]Enacted into law on July 21, 2010.

[2]Initiated in 2004 and implemented in November 2007.

[3]For example, in France, 91% of the total amount of structured funds provide a capital guarantee at maturity. Assets under management on the French retail market for structured products are equal to about 50 billion euros.

[4]As an illustration, about one third of French capital-guaranteed structured products provide a guaranteed return with a bonus based on smoothing the price variations of a reference portfolio. Products based on Asian call options can reach 14% of the guaranteed funds, while the second based on an average of past positive performances of the benchmark asset can reach 18% of the guaranteed funds.

money market accounts, they show that the customers can suffer from substantial losses when financial institutions offer a limited number of standardized portfolios which imperfectly match investor preferences.

The objective of the paper is twofold. First, we prove that standardized path dependent structured products issued by financial institutions are not optimal when considering main decision criteria, such as the expected utility maximization and one of its extensions in the portfolio insurance framework, namely the regret/rejoice criterion introduced by Bell (1982, 1983) and Loomes and Sugden (1982, 1987). For this purpose, we begin by recalling optimization results within the standard concave utility framework (with or without kink in the utility function). Then we investigate also the regret/rejoice criterion for which we provide the optimal portfolio solution. This latter criterion allows to introduce for example the regret of not having chosen a whole investment on the risk free asset if the structured product finally provides only the guarantee, which is an important issue with portfolio insurance. The non optimality of the standardized path dependent structured products is true even if the financial market is driven by a geometric Brownian motion or a usual Markov diffusion process. Second, using the notion of compensating variations, our numerical results show that the monetary losses resulting from not having access to the optimal structured portfolio can be substantial. This is in particular significant for investors who regret not having invested on the risk free asset when they get a return smaller than the risk-free one (the so-called "cash-lock risk"). This is also an important element when taking account of volatility randomness, what these standardized products do not do.

Our results are in line with previous literature about the standard portfolio allocation problem which also emphasizes the added value when introducing volatility indices exposures in portfolios, including for example VIX call options. Szado (2009) emphasizes for instance the diversification benefits of such strategy based on VIX for institutional investors (but illustrated only for a static allocation and during the 2008 crisis). Using the mean-variance analysis, Chen *et al.* (2010) show that introducing short VIX futures exposures into portfolios enhances the efficient frontier. Looking at pension funds, Warren (2012) also proves that VIX futures improve the in-sample performance of equity portfolios. Going beyond the standard mean-variance framework which can be unsuitable given the statistical characteristics of the VIX, Bahaji (2016) illustrates the attractiveness of VIX derivatives for risk-averse investors. In this paper, we show the high interest of VIX instruments for the optimal management of financial structured portfolios by measuring for the first time the monetary loss of buying standardized portfolios (which do not take account of the volatility randomness) instead of getting the optimal ones. Indeed, our numerical results emphasize that such kind of monetary loss can be higher than the corresponding risk free return on the given portfolio management period. It implies in particular that options written on volatility should be introduced in structured portfolios issued by financial institutions. From an operational point of view, they should trade on volatility indices, for example on the volatility index VIX for the S&P 500 index and the VSTOXX indices based on EURO STOXX 50, and/or should introduce options written on them in the structured portfolios.

The paper is organized as follows. Section 2 is devoted to the modelling of the financial market when taking account of the stochastic volatility of the risky asset. It provides also the analytical expressions of optimal portfolios corresponding to various assumptions about the expected utility maximization criterion. In Section 3, we examine the optimality of main path-dependent structured products issued by the financial institutions (i.e. the standardized funds). We prove that they do not correspond to the optimal solutions. Then, we introduce the notion of compensating variation to compute the monetary losses of not providing the optimal portfolios to the customers. We illustrate numerically our theoretical results by considering various values for both the risk aversion and financial parameters. Several technical details are relegated in the Appendix (see also "Supplementary Materials" for additional figures and tables).

2 The financial market model and portfolio optimization

2.1 The financial market

In what follows, we consider a financial market modelling as in Pham and Quenez (2001). It encompasses main stochastic volatility models, such as the Hull and White (1987), Heston (1993) and Black, Derman and Toy (1990) models together with the more simple standard Markovian diffusion case.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ with T a fixed maturity date and $\mathcal{F}_T = \mathcal{F}$. Recall that $\mathcal{F}$ corresponds to the dynamic available information from the financial market. Consider a financial market with a risk free asset B with instantaneous rate r and with a risky asset the dynamics of which is defined as follows:

$$\begin{aligned} dS_t &= S_t \left[\mu_t dt + \sigma(t, S_t, Y_t) dW_t^{(1)}\right], \quad (1) \\ dY_t &= m_t dt + l(t, S_t, Y_t) dW_t^{(1)} + h(t, S_t, Y_t) dW_t^{(2)}, \quad (2) \end{aligned}$$

where $W^{(1)}$ and $W^{(2)}$ are two independent standard Brownian motions under the historical probability $\mathbb{P}$. We denote by S_0 and Y_0 the initial values of previous processes. Both processes μ_t and m_t are supposed to be adapted to the filtration $\mathcal{F}$. The three deterministic functions $\sigma(t, s, y)$, $l(t, s, y)$ and $h(t, s, y)$ are measurable mappings from $[0, T] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. They are assumed to satisfy the standard conditions to ensure the existence and the uniqueness of the solution to the system (1) of stochastic differential equations, namely:[5]

1) H1: The functions $\sigma(t, s, y)$, $l(t, s, y)$ and $h(t, s, y)$ are Lipschitz in $(s, y) \in \mathbb{R}^+ \times \mathbb{R}$ uniformly in $t \in [0, T]$;

2) H2: The functions $\sigma(t, s, y)$ and $h(t, s, y)$ are non negative on $[0, T] \times \mathbb{R}^+ \times \mathbb{R}$.

We deduce that the dynamics of the volatility σ has the following form:

$$d\sigma(t, S_t, Y_t) = \sigma(t, S_t, Y_t) \left[a_t^{(\sigma)} dt + b^{(\sigma)}(t, S_t, Y_t) dW_t^{(1)} + c^{(\sigma)}(t, S_t, Y_t) dW_t^{(2)}\right]. \quad (3)$$

We assume that the function $c^{(\sigma)}(t, s, y)$ is non negative on $[0, T] \times \mathbb{R}^+ \times \mathbb{R}$.

2.2 Portfolio optimization

In what follows, we consider that both processes S and Y are observable, which is equivalent to the observation of both Brownian motions $W^{(1)}$ and $W^{(2)}$ under the previous assumptions. This yields to a complete financial market, meaning that any contingent claim is perfectly hedged from trading on the two basic assets S and σ. Such assumption is obviously satisfied when the process σ has not to be introduced (i.e. for the standard Markovian case). From the practical point of view, such assumption assumes that the portfolio manager can invest in financial products such as the volatility indices.[6]

We investigate three kinds of decision criterion: The first one corresponds to the standard expected utility maximization with a regular and concave utility function; the second one introduces a kink in the

[5] See e.g. Jacod and Shiryaev (2003) for details about these conditions.

[6] If only the risky price process S is observed and tradable, then the financial market is incomplete. In such a case, we can determine optimal portfolios using results as in Pham and Quenez (2001). Roughly speaking, the portfolio optimization problem is solved in a complete market but for a modified dynamics of the risky asset. To simplify the whole presentation of the present paper, we omit this case. Note that the introduction and development of volatility indices (VIX) instruments allow to drastically reduce this market incompleteness. Additionally, our approach allows to better illustrate the interest of introducing options written on the stochastic volatility in standardized portfolios to better fit customers needs.

utility function to better take account of aversion to the loss relatively to the risk-free investment; finally, the third one introduces explicitly the regret/rejoice of not having chosen the risk-free investment.[7]

2.2.1 Computation of the optimal portfolio for the standard decision criterion

To represent the investor's preferences, consider a utility function $U(.) : \mathbb{R}^+ \rightarrow \mathbb{R}$, which is assumed to be continuous, monotonically increasing and twice differentiable. For the standard decision criterion, the utility U is also supposed to be concave, meaning that the investor is risk averse. Let denote by J the inverse function of the marginal utility, i.e. $J = (U')^{-1}$. In what follows, we first determine the (unique) risk neutral probability measure $\mathbb{Q}$. We denote by r the riskless interest rate. The probability measure $\mathbb{Q}$ is characterized by means of its Radon-Nikodym derivative process $(\eta_t)_t$. This latter one is defined by:

$$\eta_t = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\right] =$$

$$\exp\left[-\frac{1}{2}\int_0^t\left(\beta_u^{(1)}\right)^2 du - \int_0^t \beta_u^{(1)} dW_u^{(1)} - \frac{1}{2}\int_0^t\left(\beta_u^{(2)}\right)^2 du - \int_0^t \beta_u^{(2)} dW_u^{(2)}\right], \tag{4}$$

where the premium processes $\beta_u^{(1)}$ and $\beta_u^{(2)}$ are given by:[8]

$$\beta_u^{(1)} = \frac{\mu_u - r}{\sigma(u, S_u, Y_u)} \text{ and } \beta_u^{(2)} = \frac{a_u^{(\sigma)} - r - b^{(\sigma)}(u, S_u, Y_u)\beta_u^{(1)}}{c^{(\sigma)}(u, S_u, Y_u)}. \tag{5}$$

Proposition 1 *The optimal portfolio value is given by:*

$$V_T^* = J(\lambda\eta_T), \tag{6}$$

where the parameter λ is determined from the budget constraint, namely:

$$e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[V_T^*\right] = V_0.$$

Proof. We apply results of Cox and Huang (1989). ■

Remark 2 *Looking at Formulas (6) and (4), the optimal portfolio value is function of both the whole paths of the risky asset process and the volatility process.*

2.2.2 Computation of the optimal portfolio with kink

As illustrated in Ben-Akiva *et al.* (2008), the investor may exhibit a kink in her utility function. For example, instead of having a CRRA utility function, the relative risk aversion may take two different values. In the portfolio insurance framework, such a kink can occur when the investor compares her portfolio return to the riskless rate. It corresponds to an aversion of getting a return smaller than the risk-free one.

[7] We could consider as well the Kahneman and Tversky's framework based on loss aversions. However, since we deal with portfolio insurance, the investor does not suffer from losses. Instead, she is more sensitive to comparisons with the risk free strategy.

[8] See Appendix 1 in Supplementary Materials 1.

Proposition 3 *When there is a kink in the utility function, the optimal portfolio is defined from the following equality:*

$$V_T^* = U_+'^{-1}(\lambda^* \eta_T), \tag{7}$$

where $U_+'^{-1}$ denotes the inverse of the right derivative of the utility function.

Proof. This is a particular case of Cox and Huang (1989) result. ■

To provide an illustration of such a kink, consider the modified CRRA utility defined as follows:

$$U(V) = \begin{cases} \frac{V^{(1-\gamma_1)}}{(1-\gamma_1)} & \text{if } V \geq L \\ c\frac{V^{(1-\gamma_2)}}{(1-\gamma_2)} & \text{if } V < L \end{cases},$$

with $c = \frac{(1-\gamma_2)}{(1-\gamma_1)} L^{(\gamma_2 - \gamma_1)}$ and either $0 < \gamma_2 < \gamma_1 < 1$ or $1 < \gamma_1 < \gamma_2$.

We get:

$$U_+'(V) = \begin{cases} V^{-\gamma_1} & \text{if } V \geq L \\ cV^{-\gamma_2} & \text{if } V < L \end{cases}, \text{ and } U_+'^{-1}(y) = \begin{cases} y^{\left(-\frac{1}{\gamma_1}\right)} & \text{if } y < L^{-\gamma_1} \\ L & \text{if } L^{-\gamma_1} \leq y \leq cL^{-\gamma_2} \\ \frac{y}{c}^{\left(-\frac{1}{\gamma_2}\right)} & \text{if } y > cL^{-\gamma_2} \end{cases},$$

Corollary 4 *For the GBM case, where η_T has the form $\chi\ S_T^{-\kappa}$ (see Relation 35), the optimal portfolio value when there is a kink in the CRRA type utility function, is given by:*

$$V_T^* = U_+'^{-1}\left(\lambda^* \chi S_T^{-\kappa}\right)$$

$$= \left(\lambda^* \chi S_T^{-\kappa}\right)^{\left(-\frac{1}{\gamma_1}\right)} \mathbb{I}_{\left\{\left(\lambda^* \chi S_T^{-\kappa}\right) < L^{-\gamma_1}\right\}} + L\mathbb{I}_{\left\{L^{-\gamma_1} \leq \left(\lambda^* \chi S_T^{-\kappa}\right) \leq cL^{-\gamma_2}\right\}} + \left(\frac{\lambda^* \chi S_T^{-\kappa}}{c}\right)^{\left(-\frac{1}{\gamma_2}\right)} \mathbb{I}_{\left\{\left(\lambda^* \chi S_T^{-\kappa}\right) > cL^{-\gamma_2}\right\}}$$

$$= \left(\lambda^* \chi\right)^{\left(-\frac{1}{\gamma_1}\right)} S_T^{\frac{\kappa}{\gamma_1}} \mathbb{I}_{\left\{S_T > \left(\lambda^* \chi L^{\gamma_1}\right)^{1/\kappa}\right\}} + L\mathbb{I}_{\left\{\left(\lambda^* \chi L^{\gamma_2}/c\right)^{1/\kappa} \leq S_T \leq \left(\lambda^* \chi L^{\gamma_1}\right)^{1/\kappa}\right\}} + \left(\frac{\lambda^* \chi}{c}\right)^{\left(-\frac{1}{\gamma_2}\right)} S_T^{\frac{\kappa}{\gamma_2}} \mathbb{I}_{\left\{S_T < \left(\lambda^* \chi L^{\gamma_2}/c\right)^{1/\kappa}\right\}}$$

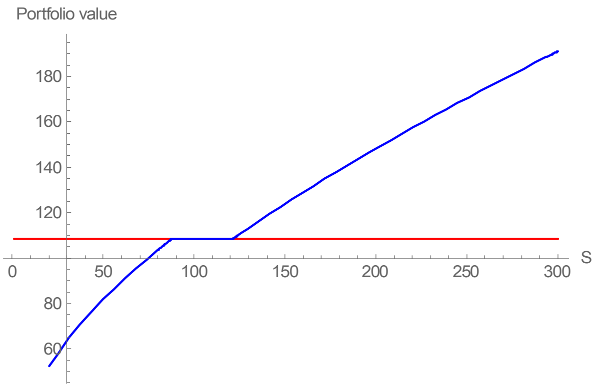
For example, consider the GBM case with the following parameter values:

$$\mu = 0.06;\ \sigma = 0.2;\ T = 8 \text{ } years;\ S_0 = 100;\ V_0 = 100; L = V_0 e^{rT}; \gamma_1 = 1.6; \gamma_2 = 2.4$$

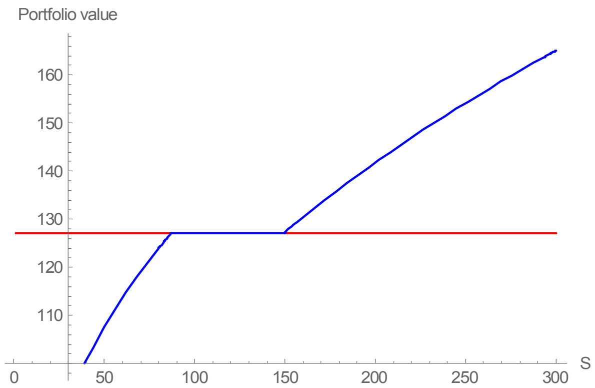
Figure 1 illustrates the optimal portfolio value as function of the risky asset value S. As expected, the payoff is equal to the constant $L = V_0 e^{rT}$ on the interval $[(\lambda^* \chi L^{\gamma_2}/c)^{1/\kappa}, (\lambda^* \chi L^{\gamma_1})^{1/\kappa}]$.

2.2.3 Computation of the optimal portfolio within the regret/rejoice framework

To go further with the notion of aversion of getting a return smaller than the risk-free one, we can base our analysis on the regret/rejoice theory.



Optimal portfolio payoff (utility with kink) $r = 1\%$



Optimal portfolio payoff (utility with kink) $r = 3\%$

Figure 1: Optimal portfolio payoff (utility with kink)

The regret/rejoice criterion Among generalizations of the standard maximization of expected concave utility functions, a well-known approach is the cumulative prospect theory with the notion of loss aversion (see Tversky and Kahneman, 1992). However, in the portfolio insurance framework, loss (after deduction of management fees) cannot exist, implying that loss aversion cannot be applied in such a context. But a well-known problem arises: Does the insured portfolio provide higher benefits than a straightforward investment on the risk free asset?[9] Similarly, the investor can regret not having sufficiently investing on the risky asset when the financial market rises. To model such attitude towards risk and return, we can involve the regret aversion as suggested by Bell (1982) and further developed by Loomes and Sudgen (1987) who emphasize that "one significant factor is an individual's capacity to anticipate feelings of regret and rejoicing."[10] Regret is also often observed when examining financial investment decisions. Using regret theory framework, Michenaud and Solnik (2005) examine usual financial problems such as international portfolio optimization. They analyze in particular the home bias equity, which corresponds to weak international diversification for investors having strong preference for domestic financial assets.[11] The regret theory supposes that individuals are «rational » but that their decisions are not only based on potential benefits (as for EU criterion) but also on expected regret with respect to other possible outcomes. Therefore, it includes two types of aversions: aversion to risk and aversion to regret.

The regret theory involves the regret or rejoice that an individual can feel when she gets outcome x instead of outcome y. The difference between these two outcomes is a measure of the rejoice or regret $r(x, y)$ defined by:

$$r(x, y) = U(x) - U(y), \tag{8}$$

[9] Bertrand and Prigent (2015 b) shows that the probability that the fund becomes monetized (known as "the cash-lock") is null for a structured fund based on the average of the past performances of the risky underlying asset while it can be significant for the OBPI or Asian type funds.

[10] This theory succeeds in explaining paradoxes such as the fanning out and the preference reversal effects (see e.g. Seidl, 2002). A large literature has been devoted to experimental psychology while a recent literature in neurobiology shows that regret has an impact on decision making under uncertainty and is not sufficiently taken into account by usual approaches (see Gilovich and Medvec, 1995; Zeelenberg, 1999; Bleichrodt and Wakker, 2015).

[11] As mentioned by Solnik (2007), the regret in investment choices leads to simple rules such as formulated by Harry Markowitz: "I should have computed the historical covariance of the asset classes and drawn an efficient frontier. Instead I visualized my grief if the stock market went way up and I wasn't in it–or if it went way down and I was completely in it. My intention was to minimize my future regret, so I split my [pension scheme] contributions 50/50 between bonds and equities."(Harry Markowitz, as quoted in Zweig (1998).).

which is negative if the individual feels regret and positive if she feels rejoice. More generally, Loomes and Sugden (1982) and Bell (1982) introduce the following modified version of the utility of x in the presence of y:

$$\mathcal{V}(x,y)=U(x)+f\left[U(x)-U(y)\right], \tag{9}$$

with $f\left[0\right]=0$, f increasing, concave and $f'''>0$.

Optimal portfolio within regret/rejoice utility In what follows, the utility U of the investor is supposed to be increasing and piecewise differentiable. The regret function f satisfies: $f\left[0\right]=0$, f is increasing, f''concave and $f'''>0$. We assume that the marginal utility U' is invertible, and we denote its inverse by J. When the investor may regret or rejoice of not having received the reference terminal wealth $\widetilde{V}_T$, the optimal payoff V_T^* is solution of the following problem:

$$Max_{V_T}\mathbb{E}_{\mathbb{P}}[U(V_T)+f\left(U\left(V_T\right)-U\left(\widetilde{V}_T\right)\right)] \text{ with } V_0=e^{-rT}\mathbb{E}_{\mathbb{P}}[V_T\eta_T].$$

Suppose that the function $\Phi_y\left[v\right]$ defined by:

$$\Phi_y\left[z\right]=U'\left[v\right]\left(1+f'\left[U\left[v\right]-U\left[y\right]\right]\right),$$

has an inverse for any value of y. Under previous assumptions on utility U and regret function f, the function Φ_y is strictly decreasing.

Proposition 5 *The optimal portfolio in the regret/rejoice framework is given by*

$$V_T^*=\Phi_{\widetilde{V}_T}\left[\lambda\eta_T\right], \tag{10}$$

where λ corresponds to the Lagrange multiplier associated to the budget constraint.
Proof. *See Appendix 2 in Supplementary Materials 1.* ■

When the terminal wealth $\widetilde{V}_T$ is a function of the terminal value of the risky asset price, namely $\widetilde{V}_T=h_0(S_T)$, we introduce the function Φ_{h_0} which is defined by:

$$\Phi_{h_0(s)}(v)=U'\left(v\right)\left[1+f'\left[U\left(v\right)-U\left(h_0(s)\right)\right]\right].$$

It is invertible as function of v.

Corollary 6 *In the GBM framework, we deduce that the optimal portfolio value is equal to the payoff $h_{RU}^*(S_T)$ given by:*

$$h_{RU}^*=\Phi_{h_0(s)}^{-1}(\lambda_{RU}\ g), \tag{11}$$

where λ_{RU} is the scalar Lagrange multiplier such that:

$$V_0e^{rT}=\int_{\mathbb{R}+}\Phi_{h_0(s)}^{-1}(\lambda_{RU}g(s))g(s)f_{S_T}(s)ds.$$

Example 7 *Consider the following regret function, as suggested in Bell (1983):*

$$f(x)=\frac{1-\exp[-ax]}{a}, \textit{ with } a>0.$$

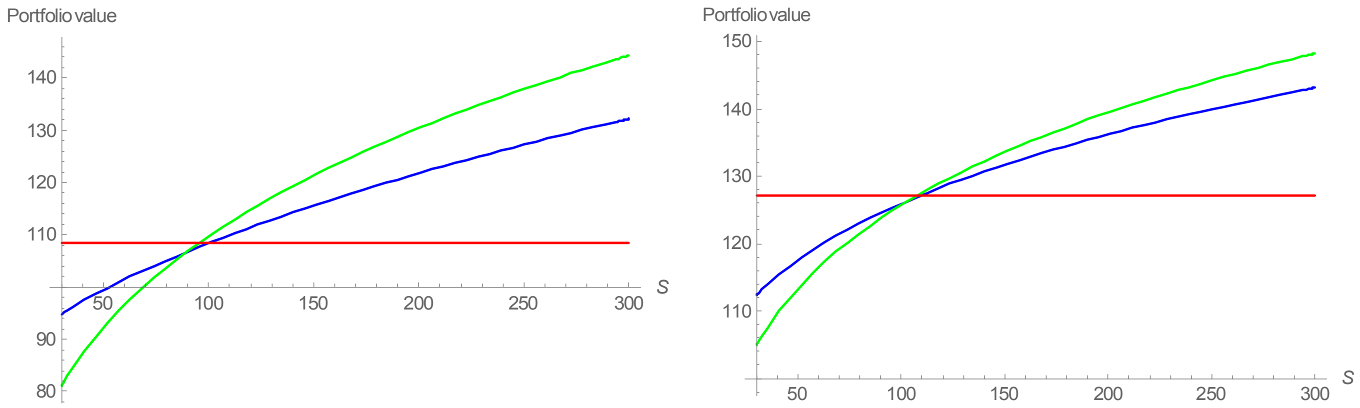
Then, for a CRRA utility $U(v)=v^{1-\gamma}/(1-\gamma)$, we have:

$$\Phi_{h_0(s)}(v)=v^{-\gamma}\left(1+\exp\left(-a\left[\frac{v^{1-\gamma}}{1-\gamma}-\frac{h_0(s)^{1-\gamma}}{1-\gamma}\right]\right)\right).$$

For example, consider the GBM case with the following parameter values:

$$\mu = 0.06;\ \sigma = 0.2;\ T = 8\ years;\ S_0 = 100;\ V_0 = 100; \gamma = 2; a = 20. \tag{12}$$

Figure (2) illustrates the optimal portfolio value as function of the risky asset value S. As expected, since we have considered here a rather high regret aversion, the optimal payoff for the regret/rejoice utility is much more close to the payoff corresponding to the risk free investment (horizontal line in the figure).



Optimal portfolio payoff (regret utility) $r = 1\%$ Optimal portfolio payoff (regret utility) $r = 3\%$

Figure 2: Optimal portfolio payoff (regret utility)

2.2.4 Computation of the optimal insured portfolio

Since we want to examine the optimality of funds in the portfolio insurance framework, we have to introduce specific guarantee constraints. The following proposition provides the optimal insured portfolio for all the previous cases. Using results of Bertrand *et al.* (2001) and El Karoui *et al.* (2005), we get:

Proposition 8 *The optimal insured portfolio corresponds to the maximum of a portfolio value without guarantee constraint and the guarantee itself. When this guarantee is equal to the initial portfolio value, we get:*

$$V_T^{**} = Max\left[J(\lambda \eta_T), V_0\right]. \tag{13}$$

Remark 9 *We note that the previous optimal solution can be viewed as a combination of an investment on the risk free asset and on a call option written on a portfolio value without guarantee constraint since we have:*

$$V_T^{**} = V_0 + Max\left[J(\lambda \eta_T) - V_0, 0\right]. \tag{14}$$

Relation (13) is true for all cases that we investigate in the paper. In Section 3, this optimal solution is compared with the standardized financial products.

3 Optimality of standardized path-dependent structured products

In this section, we examine whether or not the standardized structured products issued by financial institutions to smooth risky market fluctuations are optimal with respect to usual decision criteria. we introduce the notion of compensating variation to determine the monetary losses of the customers when financial institutions do not provide their optimal portfolios. We examine two main types of such products: the first one corresponds to a combination of a riskless investment (providing the guarantee) with an Asian call option which yields part of the positive performance of the average of the benchmark asset values along the management period; the second one is based on an average of past positive performances of the benchmark asset. We also consider the most popular insured portfolio, namely the portfolio corresponding to the *Option Based Portfolio Insurance* (OBPI), introduced by Leland and Rubinstein (1976).[12] The portfolio is invested in a risky benchmark asset S covered by a listed put written on it. The strike K of the put is equal to a predetermined proportion of the initial investment. This amount corresponds to the capital that is insured at maturity whatever the value of S at the terminal date T.[13]

3.1 The standardized structured products

3.1.1 The OBPI portfolio

The OBPI method consists basically in purchasing an amount invested on the money market account and q shares of European call options written on asset S with maturity T and exercise price K, usually the initial one S_0. The payoff is defined as follows. At maturity, the payoff of this fund is defined by:[14]

$$V_T = V_0 + \alpha_{obpi} V_0 (S_T - S_0)^+. \tag{15}$$

We denote by $Call\,(T, S_0, K, r)$ the price of each call option with maturity T, strike K, underlying S, and riskless interest rate r. We deduce that, at time 0, the budget constraint is given by:

$$V_0 = V_0 e^{-rT} + \alpha_{obpi} V_0 Call\,(T, S_0, K, r)\,. \tag{16}$$

In what follows, we set $K = S_0$. The parameter α_{obpi} is chosen such that the initial price V_0 solves equation (16):

$$\alpha_{av} = \frac{\left(1 - e^{-rT}\right)}{Call\,(T, S_0, S_0, r)}. \tag{17}$$

Therefore, the return of the OBPI strategy over the period $[0, T]$ is given by:

$$\frac{V_T}{V_0} = 1 + \left(1 - e^{-rT}\right) \frac{(S_T - S_0)^+}{Call\,(T, S_0, S_0, r)}.$$

3.1.2 Guaranteed funds based on average of positive performances

This financial structured product is based on the average of past positive performances with respect to a given underlying asset value, usually the initial one S_0. Usually, the performances (monthly, quarterly or half-yearly) are defined by averages of calls with strike equal to the initial value S_0 of the underlying

[12] See Bertrand and Prigent (2005, 2011) for more details about portfolio insurance methods and their comparison.

[13] Equivalently, a call option can be bought on S with strike K and cash held equal to the discounted value of K.

[14] Notation: $x^+ = max(x, 0)$.

risky asset. In what follows, n corresponds to the number of dates per year (for example; $n = 12, 4$ or 2), T is the time horizon (number of years) and S_{t_i} denotes the value of the risky asset at time $t_i = i/n$. The payoff of a guaranteed fund based on average (excess) performances is defined as follows. At maturity, the payoff of this fund is defined by:

$$V_T = V_0 + \alpha_{av} V_0 \frac{1}{nT} \sum\nolimits_{i=1}^{nT} (S_{t_i} - S_0)^+. \tag{18}$$

Therefore, only the S_{t_i} values of the underlying index which are higher than the initial S_0 value are taken into account in the formula.

We denote by $Call\,(t_i, S, K, r)$ the price of each call option with maturity t_i, strike K, underlying S, and riskless interest rate r. We deduce that, at time 0, the budget constraint is given by:

$$V_0 = V_0 e^{-rT} + \alpha_{av} V_0 \frac{1}{nT} \sum\nolimits_{i=1}^{nT} e^{-r(T-t_i)} Call\,(t_i, S, S_0, r)\,. \tag{19}$$

Thereafter, we denote:

$$Call^{Average}\,(T, S, S_0, r) = \frac{1}{nT} \sum\nolimits_{i=1}^{nT} e^{-r(T-t_i)} Call\,(t_i, S, S_0, r)$$

The parameter α_{av} is chosen such that the initial price V_0 solves equation (19):

$$\alpha_{av} = \frac{\left(1 - e^{-rT}\right)}{Call^{Average}\,(T, S, S_0, r)}. \tag{20}$$

The return of the average strategy over the period $[0, T]$ is given by:

$$\frac{V_T}{V_0} = 1 + \left(1 - e^{-rT}\right) \frac{\frac{1}{nT} \sum_{i=1}^{nT} (S_{t_i} - S_0)^+}{Call^{Average}\,(T, S, S_0, r)}.$$

3.1.3 Guaranteed funds based on Asian options

Asian options were introduced to take account of the average of the underlying asset values, unlike the European-type options linked to the standard OBPI method which only use value at maturity. It is not surprising, therefore, that they are being used with financial structured funds, particularly insured funds. A guaranteed fund based on discrete-time Asian options delivers a payoff defined by:

$$V_T = V_0 + \alpha_{as} V_0 \left[A\,(T) - S_0\right]^+, \tag{21}$$

with $A\,(T) = \frac{1}{nT} \sum_{i=1}^{nT} S_{t_i}$.

The term $\left[A\,(T) - S_0\right]^+$ in expression (21) represents the payoff of a discrete arithmetic Asian Call on the index with a strike price equal to the initial price S_0 of the index. We denote its price by $Call^{Asian}\,(T, A\,(T), S_0, r)$ where r corresponds to the riskless interest rate. Therefore, at time 0, the budget constraint is given by:

$$V_0 = V_0 e^{-rT} + \alpha_{as} V_0 Call^{Asian}\,(T, A\,(T), S_0, r)\,, \tag{22}$$

from which we deduce that the parameter α_{as} must be chosen such that:

$$\alpha_{as} = \frac{\left(1 - e^{-rT}\right)}{Call^{Asian}\left(T, A\left(T\right), S_0, r\right)}. \tag{23}$$

Note that the return of the strategy over the period $[0, T]$ is also defined by:

$$\frac{V_T}{V_0} = 1 + \left(1 - e^{-rT}\right) \frac{\left[A\left(T\right) - S_0\right]^+}{Call^{Asian}\left(T, A\left(T\right), S_0, r\right)}. \tag{24}$$

3.2 Comparison with optimal portfolios

In what follows, we examine the optimality of the previous standardized structured products. As it can be immediately deduced, these products can only be optimal in the case of a Markovian diffusion, since they do not involve explicitly the random paths of the volatility. To make the comparison between standardized structured products and optimal portfolios, we first detail these ones in what follows.

3.2.1 Optimal portfolio in the Markovian diffusion framework

This is a particular case of the general model (1) where both functions $\mu(t, s, y)$ and $\sigma(t, s, y)$ do not depend on Y. The stock index price dynamics is given by the following stochastic process:

$$dS_t = S_t[\mu(t, S_t)dt + \sigma(t, S_t)dW_t], \tag{25}$$

where $\mu(., .)$ and $\sigma(., .)$ are deterministic functions that satisfy the usual assumptions and W denotes a standard Brownian motion with respect to a given filtration $(\mathcal{F}_t)_t$. Then, the risky asset price S is a Markovian diffusion process. One particular case is known as the CEV model for which the volatility is given by: $\sigma(t, s, y) = s^{1-\alpha}$.

It implies:

$$S_t = S_0 \exp\left[\int_0^t \left(\mu(u, S_u) - \frac{1}{2}\sigma^2(u, S_u)\right) du + \int_0^t \sigma(u, S_u) dW_u\right]. \tag{26}$$

Under the (unique) risk neutral probability measure $\mathbb{Q}$, we have:

$$S_t = S_0 \exp\left[\int_0^t \left(r - \frac{1}{2}\sigma^2(u, S_u)\right) du + \int_0^t \sigma(u, S_u) d\widetilde{W}_u\right], \tag{27}$$

where r denotes the riskless interest rate and process $\widetilde{W}$ is a Brownian motion with respect to $\mathbb{Q}$.

Note that :

$$\eta_t = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} | \mathcal{F}_t\right] = \exp\left[-\frac{1}{2}\int_0^t \left(\frac{\mu(u, S_u) - r}{\sigma(u, S_u)}\right)^2 du - \int_0^t \frac{\mu(u, S_u) - r}{\sigma(u, S_u)} dW_u\right].$$

Assuming $\sigma(., .) > 0$, we get:

$$dW_t = \frac{dS_t/S_t - \mu(t, S_t)dt}{\sigma(t, S_t)}. \tag{28}$$

Therefore, we have:

Proposition 10 *For the Markovian case, the optimal portfolio value is given by:*

$$V_T^* = J(\lambda \eta_T), \tag{29}$$

with

$$\eta_t = \exp\left[\int_0^t \left[\frac{1}{2}\left(\frac{\mu(u,S_u)-r}{\sigma(u,S_u)}\right)\left(\frac{\mu(u,S_u)+r}{\sigma(u,S_u)}\right)\right] du - \left(\int_0^t \frac{\mu(u,S_u)-r}{S_u\sigma^2(u,S_u)}\right) dS_u\right].$$

As emphasized previously, the optimality of the main standardized path dependent structured products issued by financial institutions has a chance to be satisfied only in the Markovian framework. It is the reason why we detail this case under various assumptions on the drift and volatility functions.

Consider two fundamental examples of utility functions, namely the Constant Relative Risk Aversion (CRRA) case and the Constant Absolute Risk Aversion (CARA) case.

1) Assume that the utility function is a power function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with a constant relative risk aversion $\gamma > 0$ and $\gamma \neq 1$. We have: $U'(x) = x^{-\gamma}$and $J(y) = y^{\frac{-1}{\gamma}}$. Thus the optimal portfolio value is given by:

$$V_T^* = \lambda^{\frac{-1}{\gamma}} \exp\left[\frac{-1}{\gamma}\int_0^T \left[\frac{1}{2}\left(\frac{\mu(u,S_u)-r}{\sigma(u,S_u)}\right)\left(\frac{\mu(u,S_u)+r}{\sigma(u,S_u)}\right)\right] du + \frac{1}{\gamma}\int_0^T \left(\frac{\mu(u,S_u)-r}{S_u\sigma^2(u,S_u)}\right) dS_u\right] \tag{30}$$

2) Assume that the utility function is an exponential $U(x) = e^{-ax}/(-a)$ for $x \geq 0$ with a constant absolute risk aversion $a > 0$. We have: $U'(x) = e^{-ax}$ and $J(y) = -\frac{1}{a}Ln(y)$. Thus the optimal portfolio value is given by:

$$V_T^* = -\frac{1}{a}\left[Ln(\lambda) + \int_0^T \left[\frac{1}{2}\left(\frac{\mu(u,S_u)-r}{\sigma(u,S_u)}\right)\left(\frac{\mu(u,S_u)+r}{\sigma(u,S_u)}\right)\right] du - \int_0^T \left(\frac{\mu(u,S_u)-r}{S_u\sigma^2(u,S_u)}\right) dS_u\right] \tag{31}$$

Optimal portfolio under specific assumptions on the modified Sharpe ratio In the Markovian framework, usual discussions are based on the following modified Sharpe ratio $\frac{\mu(u,S_u)-r}{\sigma^2(u,S_u)}$. We investigate two main cases: the first one assumes that this modified Sharpe ratio is constant; the second one supposes that this ratio divided by the risky asset price is constant (i.e. $\frac{\mu(u,S_u)-r}{S_u\sigma^2(u,S_u)}$). We examine what happens for the two standard utility functions.

Assume now that the modified Sharpe ratio is equal to a constant $\widetilde{c}$, which is a common assumption in the theoretical literature about portfolio choice. It implies that the Sharpe ratio $\frac{\mu(u,S_u)-r}{\sigma(u,S_u)}$ is increasing with the volatility.

Corollary 11 *When the modified Sharpe ratio is constant, the optimal portfolio value for the CRRA case is given by:*

$$V_T^* = \lambda^{\frac{-1}{\gamma}} \exp\left[\frac{-1}{\gamma}\int_0^T \left[\frac{1}{2}\widetilde{c}\left(\mu(u,S_u)+r\right)\right] du\right] \exp\left[\frac{\widetilde{c}}{2\gamma}\int_0^T \sigma^2(u,S_u)du\right] \times S_T^{\frac{\widetilde{c}}{\gamma}}.$$

For the CARA case, it is given by:

$$V_T^* = -\frac{1}{a}\left[Ln(\lambda) + \int_0^T \left[\frac{1}{2}\widetilde{c}\left(\mu(u,S_u)+r\right)\right] du\right] + \frac{\widetilde{c}}{a}\left(\ln\left[\frac{S_T}{S_0}\right] + \frac{1}{2}\int_0^T \sigma^2(u,S_u)du\right).$$

Proof. For the CRRA case, applying Relation (30), we get:

$$V_T^* = \lambda^{\frac{-1}{\gamma}} \exp\left[\frac{-1}{\gamma}\int_0^T \left[\frac{1}{2}\widetilde{c}\left(\mu(u,S_u)+r\right)\right]du + \frac{\widetilde{c}}{\gamma}\int_0^T \frac{dS_u}{S_u}\right],$$

from which we deduce the result.

For the CARA case, using Relation (31), we get:

$$V_T^* = J(\lambda\eta_T) = -\frac{1}{a}\left[Ln(\lambda) + \int_0^T \left[\frac{1}{2}\widetilde{c}\left(\mu(u,S_u)+r\right)\right]du\right] + \frac{\widetilde{c}}{a}\int_0^T \frac{dS_u}{S_u}$$

Note that we have:

$$\int_0^T \frac{dS_u}{S_u} = \int_0^T \mu(u,S_u)du + \sigma(u,S_u)dW_u$$

$$= \int_0^T \frac{1}{2}\sigma^2(u,S_u)du \times \left[\int_0^T \left(\mu(u,S_u) - \frac{1}{2}\sigma^2(u,S_u)\right)du + \int_0^T \sigma(u,S_u)dW_u\right]$$

$$= \exp\left[\int_0^T \frac{1}{2}\sigma^2(u,S_u)du\right] \times \ln\left(\frac{S_T}{S_0}\right).$$

Therefore we deduce the result. ■

We note that, for the CRRA case, the portfolio value is a power of the risky asset value, weighted by exponentials of stochastic integrals involving respectively the drift and volatility of the risky asset. For the CARA case, the portfolio value is a linear combination of the risky asset logreturn and of temporal means of respectively the drift and volatility of the risky asset.

Assume now that the ratio $\frac{\mu(u,S_u)-r}{S_u\sigma^2(u,S_u)} = \widehat{c}$ is constant with $\mu \neq r$. We get:

Corollary 12 *When the ratio* $\frac{\mu(u,S_u)-r}{S_u\sigma^2(u,S_u)}$ *is constant, the optimal portfolio value for the CRRA case is given by:*

$$V_T^* = \lambda^{\frac{-1}{\gamma}} \exp\left[\frac{-1}{\gamma}\int_0^T \left[\frac{1}{2}\widehat{c}S_u\left(\mu(u,S_u)+r\right)\right]du + \frac{\widehat{c}}{\gamma}\left(S_T - S_0\right)\right].$$

For the CARA case, it is given by:

$$V_T^* = -\frac{1}{a}\left[Ln(\lambda) + \int_0^T \left[\frac{1}{2}\widehat{c}S_u\left(\mu(u,S_u)+r\right)\right]du + \widehat{c}S_0\right] + \frac{\widehat{c}}{a}S_T.$$

Additionally, if $\mu(u,S_u) \equiv \mu$ constant (thus $\sigma(u,S_u) = \sqrt{\frac{\mu-r}{cS_u}}$), then we deduce:

$$V_T^* = -\frac{1}{a}Ln(\lambda) - \frac{\widehat{c}}{a}S_0 - \frac{\widehat{c}\left(\mu+r\right)}{2a}\int_0^T S_u du + \frac{\widehat{c}}{a}S_T.$$

We note that, for the CRRA case, the portfolio log return is a linear combination of the risky asset value at maturity and of the temporal mean of its current values weighted by the sum of its drift with the risk free asset. For the CARA case, the portfolio value is a linear combination of the risky asset terminal value and of the temporal mean of the risky asset values.

Finally, note that the Markovian diffusion case with constant Sharpe ratio encompasses the usual geometric Brownian motion (GBM) case.[15]Due to its fundamental applications, we detail this case in what follows.

[15]Such an assumption is commonly introduced when dealing with the pricing of structured funds.

Optimal portfolio for the basic example (GBM case) Consider the risky asset dynamics defined by:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{32}$$

with constant parameters μ and σ ($\sigma > 0$). Denote:

$$\begin{aligned} m &= \mu - \frac{1}{2}\sigma^2;\ \theta = \frac{\mu - r}{\sigma} \text{ (Sharpe ratio)}, \\ a &= -\frac{1}{2}\theta^2 T + \frac{\theta}{\sigma} mT;\ \kappa = \frac{\theta}{\sigma};\ \chi = e^a (S_0)^\kappa. \end{aligned} \tag{33}$$

We consider the risk-neutral probability measure $\mathbb{Q}$ to price options. The $\sigma-$algebra $\mathcal{F}$ is generated by the Brownian motion W. We deduce that the probability density function (pdf) of $\frac{d\mathbb{Q}}{dP}$ is a function $g(S_T)$ of the terminal value of the risky asset price with respect to the σ-algebra generated by W. This function is given by:

$$g(s) = \chi s^{-\kappa}.$$

Therefore, from Relation (6), we deduce that the optimal portfolio is equal to:

$$V^*(S_T) = J\left[\lambda\ \chi\ S_T^{-\kappa}\right], \tag{34}$$

where λ is the Lagrange parameter associated to the budget constraint.

The CRRA case

Corollary 13 *For the GBM case, the optimal solution for the CRRA case is given by:*

$$V^*(S_T) = c.\chi^{-\frac{1}{\gamma}}.S_T^{\frac{\kappa}{\gamma}}, \tag{35}$$

where the power $\frac{\kappa}{\gamma}$ of S_T is equal to the Sharpe type ratio[16] $\kappa = \frac{\mu - r}{\sigma^2}$ *times the inverse of the relative risk aversion γ. The ratio $\frac{\kappa}{\gamma}$ corresponds to the Merton ratio. Applying budget constraint, the coefficient c is equal to:*

$$c = \frac{V_0 e^{rT}}{E\left[\left(\chi S_T^{-\kappa}\right)^{\frac{\gamma-1}{\gamma}}\right]} \text{ with } E\left[\left(\chi S_T^{-\kappa}\right)^{\frac{\gamma-1}{\gamma}}\right] = \exp\left(\frac{1}{2}\theta^2 T \frac{1-\gamma}{\gamma^2}\right). \tag{36}$$

Therefore, the optimal portfolio is a power of the terminal risky asset value. Note that $V^*(S_T) = h^*(S_T)$ is increasing. This property is satisfied for all concave utilities, as soon as the density g is decreasing, for instance within the Black-Scholes asset pricing framework. The concavity/convexity of the optimal payoff is determined by the comparison between the relative risk-aversion γ and the ratio $\kappa = \frac{\mu - r}{\sigma^2}$, which is the Sharpe ratio divided by the volatility σ[17]:

i) h^* is concave if $\kappa < \gamma$; ii) h^* is linear if $\kappa = \gamma$; iii) h^* is convex if $\kappa > \gamma$.

[16] We call it "Sharpe type ratio" since it is equal to the Sharpe ratio when we consider the standard deviation instead the variance.

[17] See e.g. Prigent (2007).

To illustrate the previous result, consider the following numerical base case:

$$r = 3\%, \ \sigma = 20\%, B_0 = 1, S_0 = 100.$$

The initial investment is $V_0 = 100$ and the time horizon T is equal to 1 (one year). Figure 3 illustrates how the optimal portfolio profile depends on the expected instantaneous rate of return μ of the risky asset, which has a straightforward impact on the Sharpe type ratio κ.[18] When μ is smaller than r, the optimal payoff is a decreasing function of the risky asset value S_T, which leads in practice to buy put options on S; for μ higher than r, the optimal is an increasing function of the risky asset value S_T which is concave for moderate values of μ then convex for higher values. Indeed, when the expected instantaneous rate of return μ of the risky asset is sufficiently high, the investor searches to better benefit from potential market rises.

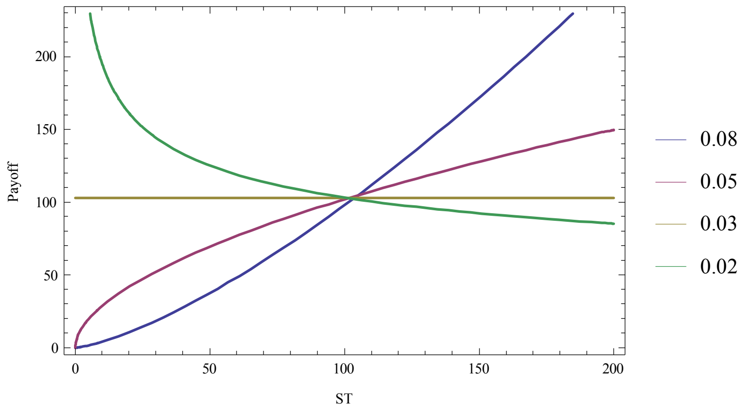


Figure 3: The optimal payoff for various values of μ

The CARA case

Corollary 14 *For the GBM case, the optimal solution for the CARA case is given by:*

$$V_T^* = -\frac{1}{a} Ln(\lambda) - \frac{c}{a} S_0 - \frac{c\,(\mu + r)}{2a} \int_0^T S_u du + \frac{c}{a} S_T.$$

Therefore, the optimal portfolio is a combination of the terminal risky asset value and of the average of its values over the time period.

[18] For this example, the relative risk aversion is fixed at the level $\gamma = 0.9$.

3.2.2 Optimality of the OBPI

The standard OBPI fund with no cap can be optimal but under very strong assumptions (GBM case and specific parameter values). Using previous results (see Corollary 13 and Proposition 8), we get:

$$V_T = V_0 + (dS_T^{\kappa/\gamma} - V_0)^+, \text{ with } d \text{ constant.} \tag{37}$$

For $\kappa/\gamma = 1$, we get the OBPI fund.[19] But recall that this fund is not used to smooth the volatility along the management period.

3.2.3 Optimality of the guaranteed fund based on average of positive performances

To examine the optimality of the guaranteed fund based on average of positive performances, we can introduce its continuous time version, namely:

$$\frac{V_T^c}{V_0} = 1 + \left(1 - e^{-rT}\right) \frac{\int_0^T (S_t - S_0)^+ dt}{Call^{c,Average}\left(T, S, S_0, r\right)}.$$

Let us examine the optimality of such a product.[20] Due to terms $(S_i - S_0)^+$ or equivalently $(S_t - S_0)^+$ in continuous time, this portfolio cannot be optimal when compared with previous solutions for the Markovian diffusion case. First, the volatility would be constant; second it would be assumed that the investor searches a portfolio optimization for each time subperiod. For example, assume that the investor maximizes her expected utility on each subperiod $[t_{i-1}, t_i]$. Assume also that she wants a guarantee on each subperiod (i.e. $V_{t_i} \geq V_0$).

Then, the optimization problem is the following:

$$Max\ \mathbb{E}\left[U_i\left(V_i\right)\right] \text{ under } V_{t_i} \geq V_0,$$

with $U_i(v) = v^{(1-\gamma)}/(1-\gamma)$.

Using previous results (see Corollary 13 and Proposition 8), we get:

$$V_{t_i} = V_0 + V_0(a_i \left(S_{t_i}/S_{t_{i-1}}\right)^{\kappa/\gamma} - 1)^+.$$

If the investor splits her initial investment into n equal amounts $V_{0,n}$ for all the subperiods with condition (i.e. $V_{0,n} = V_0/n$), then we have:

$$V_{i,n} = V_0/n + V_0/n(a_i \left(S_{t_i}/S_{t_{i-1}}\right)^{\kappa/\gamma} - 1)^+.$$

---

[19] Consider the HARA utility case:

$$U_\gamma(v) = (v - v^*)^{(1-\gamma)} / (1-\gamma),$$

where v^* corresponds to the initial invested wealth V_0. The optimal portfolio has the following form: $V_T = V_0 + \widetilde{d}S_T^{\kappa/\gamma}$, with $\widetilde{d}$ constant.

It corresponds to another portfolio insurance method, namely the Constant Proportion Portfolio Insurance (CPPI) method. As opposed to the Option Based Portfolio Insurance (OBPI) strategy, the CPPI portfolio payoff does not truly include an option whereas, for the OBPI method, the strike corresponds to the initially invested amount which insures the capital at maturity). In the paper, we only consider structured products which provide capital protection while allowing investors to benefit from market rises through more or less sophisticated derivatives written on a risky asset. The payoff at maturity is characterized by a given formula that defines the optional component.

[20] In the GBM framework, Shackleton and Wojakowski (2007) provides the explicit pricing formula of $Call^{c,Average}\left(T, S, S_0, r\right)$.

Thus, the global portfolio value at maturity is given by:

$$V_T = \sum_{i=1}^{n} V_{i,n} = V_0 \left[1 + \frac{1}{n} \sum_{i=1}^{n} (a_i \left(S_{t_i}/S_{t_{i-1}} \right)^{\kappa/\gamma} - 1)^+ \right].$$

Thus, the portfolio value at maturity is a sum of call options written on the return of the underlying asset S. However there are several shortcomings in order this fund to be truly optimal: first, this fund does not correspond to returns S_{t_i}/S_0; second, we must have $\kappa/\gamma = 1$, which is a very specific case; finally, the decision criterion is only "local" (i.e. optimization only on each subperiod).

3.2.4 Optimality of the Asian product

To examine the optimality of the Asian product, we can also introduce its continuous-time version, namely:

$$\frac{V_T^c}{V_0} = 1 + \left(1 - e^{-rT}\right) \frac{\left[\frac{1}{T}\int_0^T S_t dt - S_0\right]^+}{Call^{c,Asian}\left(T, S, S_0, r\right)}.$$

Therefore, this structured fund has no component defined on the volatility process itself. Again, for this Asian fund to be optimal, the financial market must evolve according to the Markov diffusion process described previously.

Note that, for the computation of the current value of $\mathcal{A}_T = \frac{1}{T}\int_0^T S_u du$, we can use the standard no-arbitrage valuation. Therefore, the current value of $\mathcal{A}_T$ is equal to:

$$e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[\mathcal{A}_T \left| \mathcal{F}_t \right. \right] = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{T} \int_0^T S_u du \left| \mathcal{F}_t \right. \right].$$

Since we have: for $u \geq t$, $S_t = e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}} \left[S_u \left| \mathcal{F}_t \right. \right]$, we deduce:

$$\begin{aligned} e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[\mathcal{A}_T \left| \mathcal{F}_t \right. \right] &= \frac{1}{T} e^{-r(T-t)} \left(\int_0^t S_u du + S_t \int_t^T e^{r(u-t)} du \right), \\ &= \frac{1}{T} e^{-r(T-t)} \left(\int_0^t S_u du + S_t \left[\left(e^{r(T-t)} - 1 \right) / r \right] \right). \end{aligned}$$

Let $J(y) = G \circ Ln(y)$. We search a function G such that

$$V_T^* = G \circ Ln(\lambda \eta_T) = \mathcal{A}_T,$$

This is equivalent to:

$$G(\ln \lambda + \ln \eta_T) = \mathcal{A}_T,$$

which gives:

$$G \left(\ln \lambda + \int_0^T \left[\frac{1}{2} \left(\frac{\mu(u, S_u) - r}{\sigma(u, S_u)} \right) \left(\frac{\mu(u, S_u) + r}{\sigma(u, S_{tu})} \right) \right] du - \left(\int_0^T \frac{\mu(u, S_u) - r}{S_u \sigma^2(u, S_u)} \right) dS_u \right) = \frac{1}{T} \int_0^T S_u du.$$

Denote $\frac{\mu(u,S_u)-r}{S_u \sigma^2(u,S_u)} = c(u, S_u)$.

$$G \left(\ln \lambda + \int_0^T \left[\frac{1}{2} c(u, S_u) S_u \left(\mu(u, S_u) + r \right) \right] du - \left(\int_0^T c(u, S_u) \right) dS_u \right) = \frac{1}{T} \int_0^T S_u du.$$

The only possible case corresponds to the CARA case (i.e. G is an exponential function). Looking at Corollary (14), the temporal mean $\frac{1}{T}\int_0^T S_u du$ appears in the fund's formula. However, the terminal payoff S_T is also involved.

To avoid such problem, we can also rely on the intertemporal consumption framework since the Asian product is based on temporal mean of the risky asset price. In the GBM framework, in order to get such products in the optimal portfolio, we can also introduce a consumption based model. This leads to the following expected utility:

$$\mathbb{E}\left[\int_0^T \widehat{U}(c_u, u)du + U(V_T)\right]. \tag{38}$$

Proposition 15 *Using the Hamilton-Jacobi-Bellman approach (see Karatzas et al., 1985), we deduce the optimal consumption and portfolio values.*

$$V_T^* = J(\lambda\eta_T) \text{ and } c_t^* = \widehat{J}(\lambda\eta_t).$$

Remark 16 *Consider a particular case (see Prigent, 2007, pages 198-202) for which we assume that*

$$U = 0, \widehat{U}(c) = \frac{c^{1-\gamma}}{1-\gamma} \text{ with } c > 0 \text{ and } \gamma \neq 1 \text{ and finally the Merton ratio } \frac{1}{\gamma}\frac{\mu(u, S_u) - r}{\sigma(u, S_u)^2} = 1.$$

Then, there exists only a consumption process of type $c_t^* = f(t)S_t$ where $f(t)$ is the deterministic function of time t given by:[21]

$$f(t) = e^{-a_t/\gamma} \text{ with } a_t = -\frac{1}{2}\theta^2 t + \frac{\theta}{\sigma}mt. \tag{39}$$

Thus the cumulated consumption process is based on the temporal mean of the risky asset values weighted by a deterministic function, namely $\int_0^T c_t^* dt = \int_0^T f(t)S_t dt$.

Looking at the previous optimal solution, for the Asian option to be optimal, first we must require a Merton ratio $\frac{1}{\gamma}\frac{\mu(u,S_u)-r}{\sigma(u,S_u)^2}$ equal to 1. In that case, the cumulated consumption process is based on the temporal mean of the risky asset values weighted by a deterministic function, namely $\int_0^T c_t^* dt = \int_0^T f(t)S_t dt$. Second, to get exactly the Asian payoff, the deterministic function $f(t)$ must be constant which is equivalent to $\mu(u, S_u) - r = 0$. Obviously, the two previous conditions cannot be simultaneously satisfied.

3.3 Compensating variation (monetary loss)

As shown in Bertrand and Prigent (2015b), when using Kappa performance measures (including both Omega and Sortino ratios), Asian funds are preferred to Average fund for risk-neutral investors, since the expectation of the Asian fund return is higher than the expectation of the Average fund return. But, as soon as potential risk aversion or implicitly loss aversion are taken into account, the ranking is reversed. In what follows, our goal is to go further into this comparison by first introducing various risk aversions and second to compare standardized funds with optimal ones. For this purpose, our approach relies on the notion of compensating variation.

[21] Using Relation (33).

3.3.1 Compensating variation

Several financial institutions have understood the importance to evaluate the investor's risk aversion although there are difficulties to rigorously link those measures to investment recommendations. To the best of our knowledge, one of the reasons is that those evaluations often do not provide a quantitative evaluation of investor risk aversion. Moreover, these approaches fail to describe the dynamics of risk aversion over the investor's time horizon. An exception is provided by Ben-Akiva, de Palma and Bolduc (2002) who propose an econometric method to measure quantitatively the investors' risk aversion. Typically financial institutions offer a limited number of standardized portfolios which imperfectly match investor preferences. In this paper, we quantify the efficiency losses of an investor acquiring given standardized structured funds instead of getting the portfolio corresponding to her own attitude towards risk. The compensating variation approach, introduced by Hicks (1939) in economics and by De Palma and Prigent (2008, 2009) in finance, allows to quantitatively measure the monetary loss of not receiving her own optimal portfolio.

Consider an investor with utility function parametrized by ζ and time horizon T. Denote by $V^*_{T,\zeta}$ her optimal portfolio and by V_{st} the standardized structured product sold by the financial institution. Since this latter one is generally not optimal, we search the initial value $\widehat{V}_0$ necessary to reach the same utility level. Such condition leads to the following indifference condition:

$$E[U_\zeta(V^*_{T,\zeta}; V_0)] = E[U_\zeta(V_{st}; \widehat{V}_0)]. \tag{40}$$

The ratio $\widehat{V}_0/V_0$, called the compensating variation, provides a quantitative (monetary) measure of the lack of adequacy of the standardized structured product.[22]

3.3.2 Compensating variation of the standardized path-dependent structured products

In what follows, we provide numerical examples of the compensating variations for various cases.[23] First, we focus on the standard GBM framework, for which we consider several extensions of the standard concave utility case. Second, assuming that the investor has a standard concave utility, we study the impact of the fluctuations of the stochastic volatility. For this purpose, we choose the Log Ornstein model, which is one of the main modelling of stochastic volatility.

Compensating variation for the concave utility in the GBM framework For the riskless rate, we choose two main values, namely $r = 1\%$ (low rate) and $r = 3\%$ (standard rate). Since the compensating variation is increasing with respect to r, we get its bounds for $r \in [1\%, 3\%]$, in particular for $r = 2\%$. For both these cases, we vary the volatility level of the risky asset (here, it corresponds to an equity index). For this latter purpose, we choose four main values of the volatility, namely $\sigma = 15\%$ (low volatility), $\sigma = 20\%$ (moderate volatility), $\sigma = 25\%$ (rather high volatility) and finally $\sigma = 30\%$ (high volatility). Finally, we choose two main values for the drift μ such that the risk premium $(\mu - r)$ is equal to its usual value namely 5%. [24]In what follows, our numerical base cases correspond to the following financial parameters:

$$(\mu = 6\%, r = 1\%)\,; \;\; (\mu = 8\%, r = 3\%)\,; \; T = 8; \; S_0 = 100; \; V_0 = 100.$$

[22] In De Palma and Prigent (2008), it is shown that the compensating variation can be also related to the certainty equivalent notion.

[23] We use Monte Carlo method when there is no explicit formula as in Detemple *et al.* (2003).

[24] Note also that, in Supplementary Materials 2, we examine how the compensating variation varies with respect to interest rate r (see Figures 1 to 16) and drift μ (see Figures 17 to 20). The main conclusions about the compensating variation are not significantly modified.

Figure (4) is devoted to the case $r = 1\%$ while Figure (5) corresponds to the case $r = 3\%$.

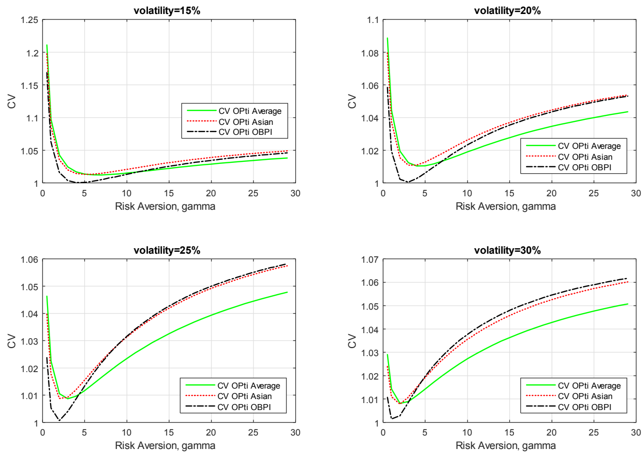


Figure 4: The CV as a function of gamma (CRRA) with GBM (mu=6% and r=1%)

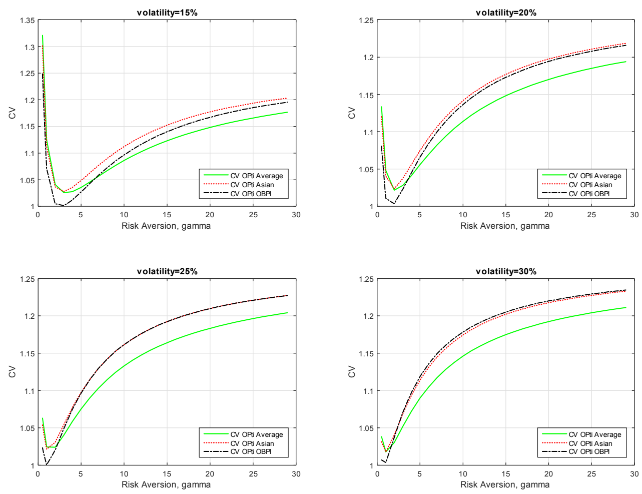


Figure 5: The CV as a function of gamma (CRRA) with GBM (mu=8% and r=3%)

Main comments:

- First, we can note that the three standardized products, namely the OBPI, the Average and the Asian funds are rather close from the compensating variation point of view. Indeed, looking at Tables 1 to 4, the maximum compensating variation is equal to 1.0327 (OBPI *versus* Average, $\gamma = 1$, $\mu = 6\%$, $\sigma = 15\%$ and $r = 1\%$). Since the fund horizon is equal to 8 years for this example, it corresponds to a monetary loss per year equal to about 0.4%;

- Second, we can notice that, as volatility increases, the OBPI is less dominant compared to the two other structured funds. For example, the Asian strategy is never dominant for $\sigma = 15\%, 20\%$ and becomes dominant only from $\sigma = 25\%$ and generally for investors having moderate or high risk aversions ($\gamma \geq 4$). It means that, according to their objective of smoothing the return of the risky asset, both Asian and Average funds are improving with respect to the standard OBPI fund when the volatility increases. Thus, as volatility increases, the performance smoothing feature of the path dependent strategies goes into action. The Average fund is generally dominant when compared to the Asian fund, especially when the volatility increases or/and the risk aversion increases. As expected, it provides a higher protection against volatility fluctuations for investors having moderate or high risk aversions ($\gamma \geq 4$).

- Third, the compensating variation between those standardized structured products and the optimal fund is very significant, compared to the compensating variations between only standardized structured products. For the compensating variation between the OBPI fund and the optimal one, it can reach for example 1.176 for $\gamma = 10$, $\sigma = 25\%$ and $r = 3\%$ meaning a monetary loss per year equal to about 2.24% (see Tables 5 to 8). This would induce an implicit managing cost higher than 2% per year! However, for both interest rate levels, namely $r = 1\%$ and $r = 3\%$, the maximum compensating variation is always smaller than the return of the risk free asset on the same time period. Such feature illustrates the lack of adequacy to customers needs, especially for both "aggressive investors" ($\gamma = 1$) in a moderate ($\sigma = 20\%$) or low ($\sigma = 15\%$) volatility environment and for "conservative" investors ($\gamma = 10$);

- Fourth, the behavior of the compensating variation as a function of risk aversion and volatility levels is rather complex. It depends on all the financial parameter values: For fixed financial parameter values, the compensating variation is a function of the risk aversion γ which can be either first decreasing then increasing or always increasing. Indeed, as it can be seen in Relation (37), the compensating variation is null when the risk aversion γ is such that the OBPI fund is optimal ($\gamma = \gamma^{OBPI}$). Thus, as soon as the distance between γ and γ^{OBPI} increases, the compensating variation does also. As it can be seen in Appendices 3 and 4 (see Supplementary Materials 1), for a simplified case with no insurance constraint, for fixed risk aversion parameter γ, the compensating variation between the optimal fund and an optimal fund related to the OBPI fund is increasing with the volatility if γ is sufficiently high. Such property is rather robust when dealing with portfolio insurance. Indeed, Relation 13 characterizes the optimal insured portfolio as the maximum between an optimal portfolio without insurance constraint and, for example, a fixed guaranteed amount. Looking at Figures (4) and (5), (see also Figures (9) to (20) in Supplementary Materials 2), we can notice such feature. Indeed, when $\sigma = 15\%$, the compensating variation between the optimal fund and the three standardized funds is higher for small values of the risk aversion γ than for the three other values of the volatility, namely $\sigma = 20\%, 25\%, 30\%$. For moderate or high risk aversion γ, it is the converse: for fixed $\gamma \geq 5$, the higher the volatility, the higher the compensating variation.

- Finally, as function of time horizon, the compensating variation is increasing, as illustrated by Figures (21), (22) and (23) (see Supplementary Materials 2). Finally, note that the choice of the utility function does not change very significantly the numerical values of the compensating variation (as explained in Supplementary Materials 2).

Compensating variation for the concave utility with kink case in the GBM framework In this case, the dynamic of the risky asset process is given by a geometric Brownian motion ($dS_t = S_t\left[\mu dt + \sigma dW_t\right]$). The utility with kink is given by:

$$U(V) = \begin{cases} \frac{V^{(1-\gamma_1)}}{(1-\gamma_1)} & \text{if } V \geq L \\ c\frac{V^{(1-\gamma_2)}}{(1-\gamma_2)} & \text{if } V < L \end{cases}, \text{ with } c = \frac{(1-\gamma_2)}{(1-\gamma_1)} L^{(\gamma_2-\gamma_1)}.$$

Looking at Tables (11) to (18) in Supplementary Materials 3, we deduce similar results as previously when comparing the three standardized products between them and with the optimal fund. We still observe that higher volatility generally implies higher compensating variations for moderate or high risk aversion γ. These latter ones are more sensitive to the volatility than to the threshold determining the kink around the riskless return. Most of the time, the compensating variation is decreasing with respect to the drift of the risky asset. We can also examine the compensating variation as function of the kink level. Recall that we choose $L = 0.75\exp[rT]$, $L = 0.75\exp[rT]$ and finally $L = 1.25\exp[rT]$.

For the case $r = 1\%$, we compare results for the standard concave case with $\mu = 6\%$ and $\sigma = 20\%$ (see Table 2) to those in Tables (11) to (14) corresponding to the kink case. We use the following approximate correspondence between the standard concave and the kink cases: For $\gamma = 2$, $\gamma_1 = 1.6$ and $\gamma_2 = 2.4$ for the utility with kink; for $\gamma = 6$, $\gamma_1 = 5.2$ and $\gamma_2 = 6.8$ for the utility with kink; for $\gamma = 10$, $\gamma_1 = 8.5$ and $\gamma_2 = 11.5$ for the utility with kink. For $\gamma = 2$, the compensating variations for the standard concave case are higher than for the kink case, while for both $\gamma = 6$ and $\gamma = 10$, it is generally the converse. Most of the time, the compensating variation is slightly increasing with respect to level of the kink L but the difference is relatively weak.

For the case $r = 3\%$, we compare results for the standard concave case with $\mu = 6\%$ and $\sigma = 20\%$ (see Table 6) to those in Tables (15) to (18) corresponding to the kink case. Most of the time, the compensating variations for the kink case is very close to that for the standard utility case. We note also that, for $\gamma = 6$ and $\gamma = 10$, the compensating variation is not sensitive to the level of the kink (at least for the three considered values, namely $L = 0.75\exp[rT]$, $L = 0.75\exp[rT]$ and $L = 1.25\exp[rT]$).

Therefore, to better investigate the impact of the aversion of getting a return smaller than the risk-free one, we analyze numerically the regret/rejoice criterion in what follows.

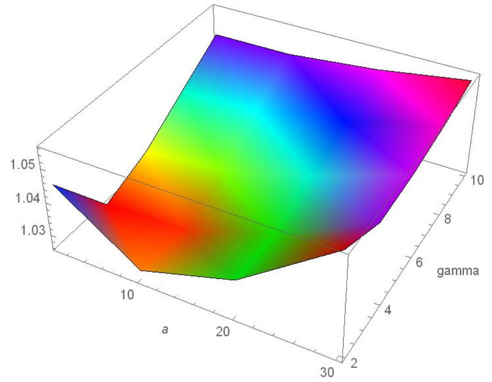
Compensating variation for the regret/rejoice case in the GBM framework To illustrate the compensating variation for the regret/rejoice case, we choose the regret function f, as suggested in Bell (1983), namely:

$$f(x) = \frac{1 - \exp[-ax]}{a}, \text{with } a > 0.$$

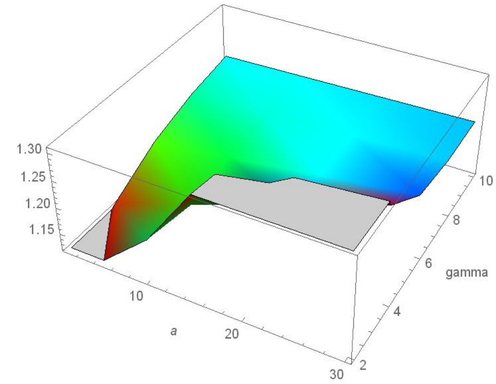
Then, we choose a CRRA utility $U(v) = v^{1-\gamma}/(1-\gamma)$. Thus, we get:

$$\Phi_{h_0(s)}(z) = z^{-\gamma}\left(1 + \exp\left(-a\left[\frac{z^{1-\gamma}}{1-\gamma} - \frac{h_0(s)^{1-\gamma}}{1-\gamma}\right]\right)\right).$$

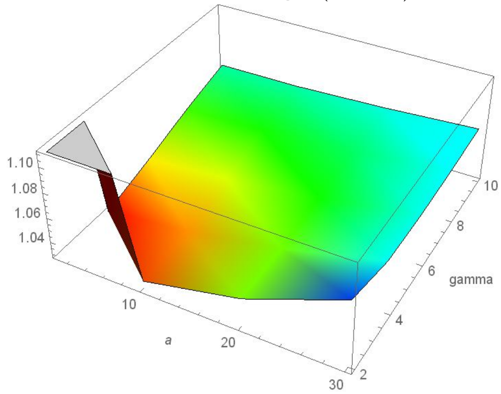
Applying results of Proposition 5, we can numerically analyze the impact of both the relative risk aversion γ and the parameter a defining the regret/rejoice function. For this purpose, we set the following financial parameters: The drift μ of the risky asset is equal to 6%; the volatility σ of the risky asset is equal to 20%; the interest rate is equal to either to 1% or to 3%. We consider also a time horizon T equal to 8 years. To calibrate the parameter a associated to the regret function, we use results in Michenaud and Solnik (2008) based on regret aversion. This latter parameter has been defined by Bell (1983) who has introduced the ratio $RA = -\frac{V_0 f'' U'}{1+f'}$. This leads us to consider the following values for the parameter a: $a = 1$, 10, 20, and finally 30. Note that these values yield to regret aversion levels RA lying mainly between 2 and 30. Note also that, when letting a converges to 0, the regret function converges to the linear function (namely $\lim_{a\to 0^+} f(x) = x$), which ultimately corresponds to the standard utility case.



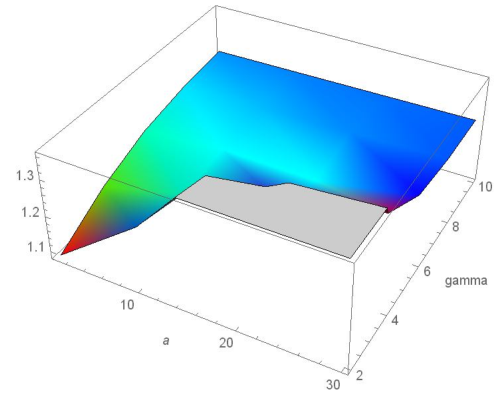
Cv Opti Average (r=1%)



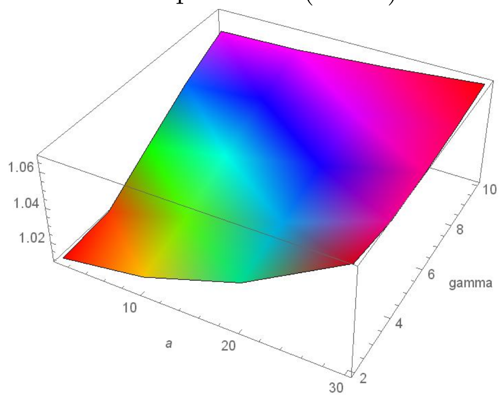
Cv Opti Average (r=3%)



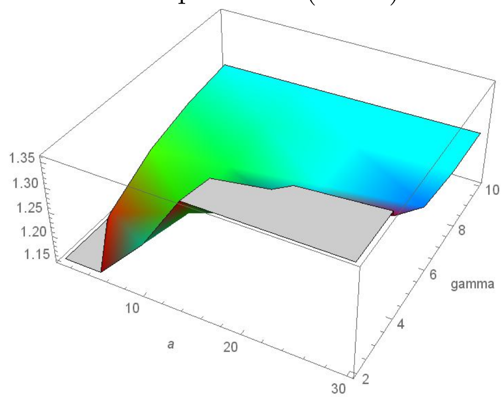
Cv Opti Asian (r=1%)



Cv Opti Asian (r=3%)



Cv Opti OBPI (r=1%)



Cv Opti OBPI (r=3%)

Figure 6: Compensating variations for utility with regret/rejoice

Figure (6) illustrates the compensating variation for utility with regret/rejoice (see also Tables (9) and (10). For the case $r = 1\%$, we compare results in Table (2) for the standard concave case with $\mu = 6\%$ and $\sigma = 20\%$. We note first that the compensating variations for the regret/rejoice case are much higher than for the standard utility case. Indeed, they can reach values equal to about 6.5% corresponding approximately to 0.8% per year, which is close to the risk-free rate itself. We note also that, except for the case $\gamma = 2$, the compensating variation for the regret/rejoice criterion is an increasing function of the parameter a associated to the regret/rejoice function. Table (9) displays the compensating variations when comparing the optimal portfolio to the three standardized structured products for the case $r = 3\%$. For the case $r = 3\%$, we compare results in Table (10) to those in Table (6) for the standard concave case with $\mu = 6\%$ and $\sigma = 20\%$. As previously, we note also that the compensating variations for the regret/rejoice case are much higher than for the standard utility case. Indeed, they can reach very high values for weak relative risk aversion such as $\gamma = 2$. Even for stronger relative risk aversions, for example $\gamma = 10$, the compensating variation for the regret/rejoice case can be equal to about 25.8% corresponding approximately to 2.87% per year, which is still close to the risk-free rate itself. We note also that here the compensating variation for the regret/rejoice criterion is always an increasing function of the parameter a associated to the regret/rejoice function.

Compensating variation for the standard concave utility case with the Log Ornstein Uhlenbeck model for stochastic volatility This is the model introduced by Black, Derman and Toy (1990). Let $\mu_t \equiv \mu$ constant and $\sigma(t, S_t, Y_t) \equiv f(Y_t)$ with $f(y) = e^y$. Here the process Y denotes the logarithm of the volatility. It is supposed to follow an Ornstein-Uhlenbeck model. In this case, the dynamics of the risky asset process is given by: (In Relation 1, it corresponds to the choice $m_t = k(Y_\infty - Y); l(t, S_t, Y_t) = \rho\xi; h(t, S_t, Y_t) = \sqrt{1-\rho^2\xi})$

$$dS_t = S_t \left[\mu dt + exp(Y_t) dW_t^{(1)}\right], \text{ with } dY_t = k(Y_\infty - Y)dt + \rho\xi dW_t^{(1)} + \sqrt{1-\rho^2}\xi dW_t^{(2)},$$

with constant parameters μ (the drift of the risky asset), $k > 0$, $\xi > 0$ (standard deviation of dY), ρ (correlation of the two Brownian motions). Thus Y is mean reverting and converges to its long term value Y_∞ with a speed equal to k. Applying Ito's lemma to $exp(Y)$, we deduce that the parameters of volatility σ defined in Relation (3) are given by: $a_t^{(\sigma)} = k(\ln\sigma_\infty - \ln\sigma) + 0.5\xi^2; b^{(\sigma)}(t, S_t, Y_t) = \rho\xi; c^{(\sigma)}(t, S_t, Y_t) = \sqrt{1-\rho^2}\xi$. The two premium processes $\beta_u^{(1)}$ and $\beta_u^{(2)}$ defined in Relation 5 are given by:

$$\beta_u^{(1)} = \frac{\mu - r}{\exp[Y_u]}, \beta_u^{(2)} = \frac{k(Y_\infty - Y_u) - r - \rho\xi\frac{\mu - r}{\exp[Y_u]}}{\sqrt{1-\rho^2}\xi}.$$

For the numerical illustration, we consider the following financial parameters:

$$(\mu = 0.06; r = 0.01);\ (\mu = 0.08; r = 0.03),\ \ k = 0.2.$$

We set $Y_\infty = Y_0 = Log[0.2]$, which allows a better comparison to the GBM case with constant volatility σ equal to 20%.[25] Indeed, by setting both the initial volatility $\sigma_0 = exp[Y_0]$ and the long term volatility $\sigma_\infty = exp[Y_\infty]$ to the same level equal to 20%, we only take account of the stochastic feature of the volatility. For this purpose, we consider four choices for the pair of parameters (ρ, ξ), namely $\rho = -0.5$ or 0, and $\xi = 0.05$ or $\xi = 0.10$. Recall that parameter ρ allows to calibrate the dependency

[25] In Supplementary Materials 4, we have conducted the same analysis for two other levels, namely $Y_\infty = Y_0 = Log[0.15]$ and $Y_\infty = Y_0 = Log[0.25]$. As shown by Figures (24) to (27), we get the same features of the compensating variation.

between the return of the risky asset S and the logarithm of its volatility (usually ρ is negative) while parameter ξ allows to consider various levels of the "volatility of the volatility."

For the case $r = 1\%$. Looking at Figure (7), we note that the three standardized products, namely the OBPI, the Average and the Asian funds, are rather close from the compensating variation point of view. Indeed, the maximum compensating variation is equal to 1.0245. Since the fund horizon is equal to 8 years for this example, it corresponds to a monetary loss per year about equal to 0.3%. These weak compensating variations between the three standardized structured products are also illustrated in Tables (19) to (21) (see Supplementary Materials 4), from which we can also deduce that parameters ρ and ξ do not have a significant impact on these compensating variations.

On the contrary, the compensating variations between those standardized structured products and the optimal fund are very significant. Compared to results in Table (6) corresponding to the standard GBM case, they are much higher. For example, they can reach 1.2113 meaning a monetary loss per year equal to about 2.43%. This induces an implicit managing cost higher than 2% per year ! We note that most of the compensating variations are close to 1.07 which is only slightly smaller than the risk fee return 1.083 on 8 years. For the GBM case, the maximum value is equal to 1.0436 corresponding "only" to 0.53% per year. Such feature illustrates the lack of adequacy of those three standardized products to customers needs when the volatility is stochastic, especially for "aggressive" investors ($\gamma = 1$ or 2). Figure (7) illustrates also the compensating variations between standardized structured products and the optimal fund. We note that we get the maximal compensating variations for the cases $\rho = 0$, $\xi = 0.05$, $\rho = -0.5$, $\xi = 0.10$ and $\rho = -0.5$, $\xi = 0.05$, while the minimal compensating variations is reached for the case $\rho = 0$, $\xi = 0.10$. Indeed, for instance when the correlation ρ is null, it is interesting to be able to use a financial instrument defined on the volatility itself but the higher its volatility the smaller the interest to diversify on it since it becomes riskier. When the correlation is negative (which is for example usually the case between a stock index and its own volatility), the higher the risk aversion from the level 4, the higher the expected utility since the investor can benefit from the diversification (see Appendix 4 in Supplementary Materials 1 for a simplified example explaining most of these features).

For the case $r = 3\%$, we choose $\mu = 8\%$ so that we get the same risk premium as in previous case, namely the standard value $\mu - r = 5\%$. We set the portfolio horizon to 5 years, which implies that the initial proportion of wealth invested on the risky asset is about 14% (it is equal to about 7.7% for the case with $r = 1\%$). Note that, from the operational point of view, the horizon equal to 5 years is the standard value used by practitioners as soon as the interest rate is about 3%. We recover the same main features, namely the compensating variations between the standardized structured products and the optimal fund are very significant, compared to those between the standardized structured products themselves. We note in particular that the compensating variations can reach levels significantly higher than the risk free return on the given time period (i.e. $exp\,[rT] \simeq 1.162$), for the cases $\rho = 0$, $\xi = 0.05$ and $\rho = -0.5$, $\xi = 0.05$, while the minimal compensating variations is reached for the cases $\rho = 0$, $\xi = 0.10$ and $\rho = -0.5$, $\xi = 0.10$. (see Figure 8). As previously explained, for $\xi = 0.05$, the Sharpe type ratio of the Log volatility is higher than for $\xi = 0.10$, implying higher compensating variations. Compared to the case $r = 1\%$ and $\mu = 6\%$, when the risk free rate is equal to 3%, the factor loading $\beta_u^{(2)}$ (see Relation 5) can have a significant smaller absolute value for $\rho = -0.5$, $\xi = 0.10$, explaining why, for this second case (i.e. $r = 3\%$ and $\mu = 8\%$) , it is less interesting to be able to trade on the volatility (see remark about condition $0 < r - \mu^{(2)}$ in Appendix 4). Obviously, when r is high, this condition has much more chance to be met). On the contrary, for the case $r = 3\%$, $\mu = 8\%$ and portfolio horizon T equal to 5 years, we can better fit from trading on the volatility when its Sharpe ratio is relatively high since the initial proportion of wealth invested on the risky asset is equal to about twice that for the case $r = 1\%$, $\mu = 6\%$ and $T = 8$.

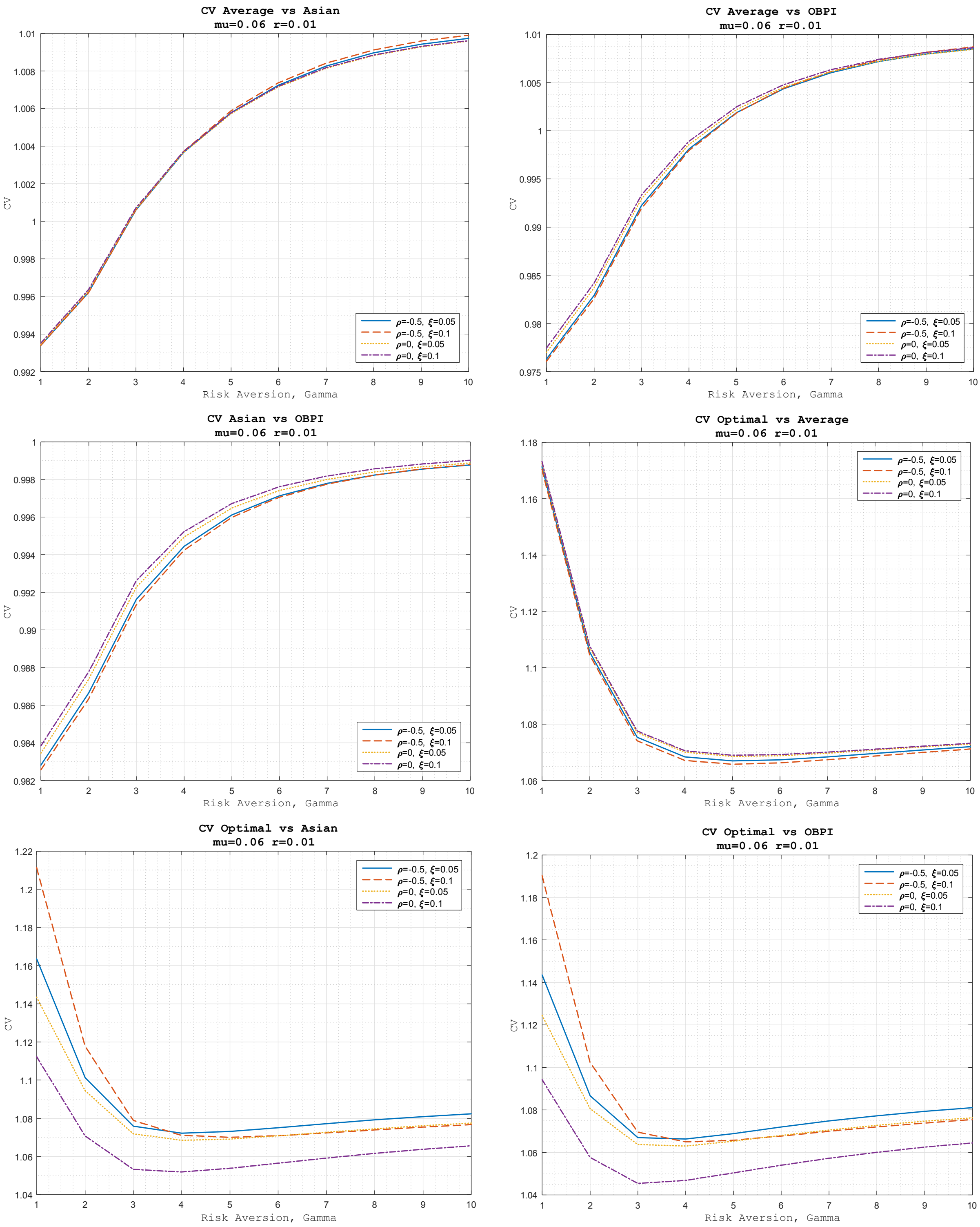


Figure 7: Compensating variations for the Ornstein-Uhlenbeck case case (mu=0.06, r=0.01)

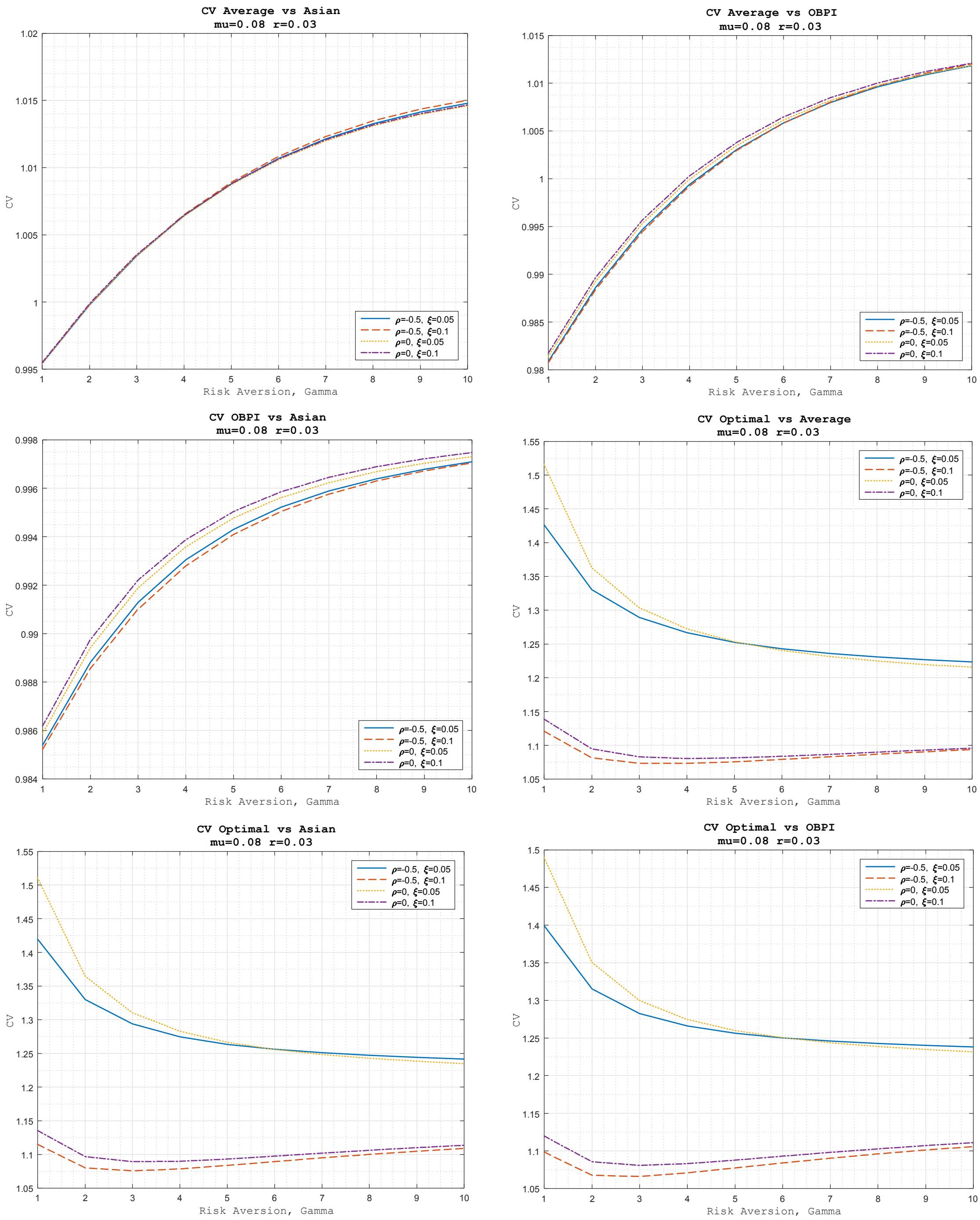


Figure 8: Compensating variations for the Ornstein-Uhlenbeck case (mu=0.08,r=0.03)

4 Conclusion

The suitability of standard financial structured products is a very important topic for the costumer protection. We look at an important class of such products, namely those whose performances are based on smoothing the return of a given risky underlying asset while providing a guarantee at maturity. We illustrate the potential inadequacy of such standardized financial funds to customers, depending on their attitudes towards risk. From the theoretical point of view, we prove that such standardized products are not optimal, even if the financial market volatility is constant. By using the notion of compensating variation, we compute the monetary losses of providing these standardized products instead of the optimal ones to the customers. We show that these (theoretical) losses can be severe, since they can be higher than the risk-free return on the given portfolio management period. It implies in particular that trading on volatility indices and/or options written on them should be introduced in structured portfolios issued by financial institutions (for example on the volatility index VIX for the S&P 500 index and the VSTOXX indices based on EURO STOXX 50). Another possible study would consist of calculating also the lowest cost strategies to achieve the given payoff distribution of each standardized fund (see e.g. the distributional analysis of portfolio choice and the theory of cost-efficiency of Dybvig (1988a, 1988b) and Bernard *et al.*, 2014).

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